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Quantization of inhomogeneous Lie bialgebras

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Abstract

A self-dual class of Lie bialgebra structures $(\mathfrak{g}, \mathfrak{g}^*)$ on inhomogeneous Lie algebras $\mathfrak{g}$ describing kinematical symmetries is considered. In that class, both $\mathfrak{g}$ and $\mathfrak{g}^*$ split into the semi-direct sums $\mathfrak{g} = \mathfrak{h} \triangleright \mathfrak{v}$ and $\mathfrak{g}^* = \mathfrak{h}^* \triangleleft \mathfrak{v}^*$ with abelian ideals of translations $\mathfrak{v}$ and $\mathfrak{h}^*$. We build the explicit quantization of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$, including the coproduct, commutation relations among generators, and, in case of coboundary $\mathfrak{g}$, the universal R-matrix. This class of Lie bialgebras forms a self-dual category stable under the classical double procedure. The quantization turns out to be a functor to the category of Hopf algebras which commutes with operations of dualization and double.

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1. Introduction

Inhomogeneous Lie groups such as Poincaré and Galilei groups play an important role in classical physics and geometry [1]. They realize the maximal sets of (continuous) symmetries of the simply connected (pseudo)Riemannian spaces of zero curvature [2]. The generalization of the semi-direct product of classical groups in the framework of non-commutative geometry is the bicrossproduct of two Hopf algebras $\mathcal{A}$ and $\mathcal{B}$ characterized by actions of $\mathcal{A}$ on $\mathcal{B}$ and $\mathcal{B}^*$ on $\mathcal{A}^*$ (the Hopf duals to $\mathcal{B}$ and $\mathcal{A}$) [4]. Nowadays, there are numerous examples of bicrossproducts known, including those among quantum deformations of the Cayley–Klein algebras [3]. The simplest case of the bicrossproduct construction is the second (so-called non-standard) quantization of the Borel subalgebra $b(2) \subset \mathrm{sl}(2)$ [5].

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This algebra is also the result of Drinfeld's twist [7,8] of the universal enveloping algebra $\mathcal{U}(b(2))$ [6]. Another examples of twisted bicrossproduct Hopf algebra are the null-plane quantized Poincaré algebra [9] and extended jordanian deformations of $\mathcal{U}(\mathrm{sl}(N))$ [10]. These quantizations involve special non-degenerate 1-cocycles on Lie groups [10–12]. All those algebras are twist-equivalent to classical universal enveloping algebras, and that equivalence manifests itself in an isomorphism of the corresponding tensor categories of modules. Quasitriangular bicrossproduct Hopf algebras with non-unitary R-matrices were found in [13] via the quantum double construction in the framework of the matched pairs of finite groups. The present work is devoted to the study of "continuous" bicrossproduct Hopf algebras with abelian invariant subalgebras. In the classical differential geometry, these correspond to inhomogeneous Lie groups, containing a normal vector subgroup of translations. Quantization of such algebras appears to possess a number of remarkable features, for example, invariance of the category of interest with respect to dualization and the double procedures. We obtain explicit expressions for the coproduct, commutation relations, and the antipode in the quantized algebras. That allowed us to carry out the detailed study of their quantum doubles, construct canonical elements and R-matrices for generic quasitriangular Hopf algebras from the category under investigation. We use a kind of "universality" of the double construction, discovered by Radford [14] and meaning the following: every quasitriangular Hopf algebra contains a minimal quasitriangular Hopf subalgebra which is a quotient of the quantum double of a certain Hopf subalgebra. We prove the Lie bialgebra analog of this theorem.

The possibility to quantize an arbitrary Poisson manifold was proven in [15]. Nevertheless, to construct such a quantization explicitly remains a very difficult task. In the case of semi-simple Lie algebras, this problem is solved for every Belavin–Drinfeld triple [16] by twisting the standard quantization of Drinfeld [5]. In this paper, we deal with a different class of Lie algebras which possess abelian ideals.

Let G and G^* be the simply connected Lie groups corresponding to the Lie algebras $\mathfrak{g}$ and $\mathfrak{g}^*$. They are decomposed into the smashed products $G = H \triangleright V$ and $G = H^* \triangleleft V^*$ of groups corresponding to the semi-direct sums $\mathfrak{g} = \mathfrak{h} \triangleright \mathfrak{v}$ and $\mathfrak{g}^* = \mathfrak{h}^* \triangleleft \mathfrak{v}^*$. The generators of the quantized universal enveloping algebras constructed below may be thought of as functions on the group G^* endowed with the Poisson–Lie structure. It turns out that the commutation relations between generators are exactly the Poisson–Lie brackets. In particular, the restriction of this bracket to the subgroup V^* is commutative and remains undeformed. This picture is similar to that in the polarized deformation quantization [17]. According to [17], the $*$-product restricted to the sheaf of functions constant along a polarization of a sufficiently general type remains undeformed.

2. Quantum double and quasitriangularity

This section formulates the analog of Radford's theorem on minimal quasitriangular Hopf algebras [14]. This result can be carried over to a coboundary Lie bialgebra whose r-matrix satisfies the classical Yang–Baxter equation [18]. We start with the following elementary proposition from the linear algebra. Let $\mathbf{L}$ be a finite dimensional vector space and $\mathbf{L}^*$ be its dual. Consider an element $r \in \mathbf{L} \otimes \mathbf{L}$, which is thought of as a linear operator

$\mathbf{L}^* \to \mathbf{L}$: $r(x) = \langle x \otimes \mathrm{id}, r\rangle$. The dual conjugate r^* is again an operator $\mathbf{L}^* \to \mathbf{L}$ acting as $r^*(x) = \langle \mathrm{id} \otimes x, r\rangle$. Denote $\mathbf{L}_+$ and $\mathbf{L}_-$ the images of r and r^*, respectively. Actually, $r \in \mathbf{L}_- \otimes \mathbf{L}_+ \subset \mathbf{L} \otimes \mathbf{L}$. Let $\{l_i\}$ be a basis in $\mathbf{L}_+$ and $\{l^i\}$ its dual $\mathbf{L}^*$. The canonical element $l = l^i \otimes l_i \in \mathbf{L}_+^* \otimes \mathbf{L}_+$ does not depend on the choice of the basis.

Lemma 2.1. *As linear spaces, $\mathbf{L}_+^* \sim \mathbf{L}_-$. The element r is the image of the canonical element l under the induced isomorphism $\mathbf{L}_+^* \otimes \mathbf{L}_+ \to \mathbf{L}_- \otimes \mathbf{L}_+$ identical on the second tensor factor.*

Proof. The first statement of the lemma follows from the two commutative diagrams:

$$\begin{tikzcd} \mathbf{L}^* \arrow[rr, "r"] \arrow[dr, "\tilde r"'] & & \mathbf{L} \\ & \mathbf{L}_+ \arrow[ur, "i"'] \arrow[dr] & \\ 0 \arrow[ur] & & 0 \end{tikzcd} \qquad \begin{tikzcd} \mathbf{L} & & \mathbf{L}^* \arrow[ll, "r^*"'] \arrow[dl, "i^*"] \\ & \mathbf{L}_+^* \arrow[ul, "\tilde r^*"] \arrow[dl] & \\ 0 & & 0 \arrow[ul] \end{tikzcd}$$

where $\tilde r$ is defined by the left one, and the isomorphism $\mathbf{L}_+^* \to \mathbf{L}_-$ is given by the dual map $\tilde r^*$. Let us prove that the map $\tilde r^* \otimes \mathrm{id}$ brings the canonical element $l \in \mathbf{L}_+^* \otimes \mathbf{L}_+$ right to r. Indeed, the canonical element is interpreted as the identity map via the identification $\mathbf{L}_+^* \otimes \mathbf{L}_+ \sim \mathrm{Hom}(\mathbf{L}_+, \mathbf{L}_+)$: $\langle l, r(x) \otimes \mathrm{id}\rangle = r(x)$. Then, for every $x, y \in \mathbf{L}^*$, we have $\langle(\tilde r^* \otimes \mathrm{id})(l), x \otimes y\rangle = \langle l, r(x) \otimes y\rangle = \langle r(x), y\rangle = \langle r, x \otimes y\rangle$. □

Now, consider the situation when $\mathbf{L}$ is a finite dimensional Lie bialgebra $\mathfrak{g}$, whose Lie cobracket is determined by a solution $r \in \mathfrak{g}^{\otimes 2}$ of the classical Yang–Baxter equation [18]

$$[r_{12}, r_{13}] + [r_{12}, r_{23}] + [r_{13}, r_{23}] = 0.$$

This equation is supported in $\in \mathfrak{g}^{\otimes 3}$, and the subscripts indicate the way of embedding of the tensor square into the tensor cube. It follows from this equation that the images $\mathfrak{g}_+ = r(\mathfrak{g}^*)$, $\mathfrak{g}_- = r^*(\mathfrak{g}^*)$, and their linear sum are themselves sub-Lie bialgebras. Moreover, $\mathfrak{g}_+ + \mathfrak{g}_-$ is the minimal quasitriangular sub-Lie bialgebra, where the classical r-matrix lives in fact. Since r^* is Lie algebra homomorphism but a coalgebra anti-homomorphism, it also can be regarded as a morphism in the Lie bialgebra category, $\mathfrak{g}_+^*$ being endowed with the opposite bracket. According to [5,18], the classical double $\mathcal{D}(\mathfrak{g}_+)$ is a unique Lie algebra on $\mathfrak{g}_+ + \mathfrak{g}_-$ such that $\mathfrak{g}_\pm$ are Lie subalgebras and the pairing between them is ad-invariant. The double $\mathcal{D}(\mathfrak{g}_+)$ is a coboundary Lie bialgebra, with the classical r-matrix being the canonical element l defined as in Lemma 2.1 for $\mathfrak{g} = \mathbf{L}$. It is considered as an element of $\mathcal{D}(\mathfrak{g}_+) \otimes \mathcal{D}(\mathfrak{g}_+)$.

Theorem 2.2. *Embeddings $\mathfrak{g}_\pm \to \mathfrak{g}$ define a homomorphism $\mathcal{D}(\mathfrak{g}_+) \to \mathfrak{g}$. The matrix $r \in \mathfrak{g}^{\otimes 2}$ is the image of the canonical element $l \in \mathcal{D}^{\otimes 2}(\mathfrak{g}_+)$ under this homomorphism.*

Proof. By Lemma 2.1, r is the image of the canonical element l. Restricted to $\mathfrak{g}_\pm$, the map $\mathcal{D}(\mathfrak{g}_+) \to \mathfrak{g}$ preserves the Lie structures separately. Let us show that for the commutator $[\mathfrak{g}_+, \mathfrak{g}_-]$. Consider the paring of the left-hand side of the classical Yang–Baxter equation with arbitrary elements $x, y, z \in \mathfrak{g}^*$

$$\langle[r_{12}, r_{13}], x \otimes y \otimes z\rangle + \langle[r_{13}, r_{23}], x \otimes y \otimes z\rangle + \langle[r_{12}, r_{23}], x \otimes y \otimes z\rangle = 0.$$

Having introduced the notations $x_+ = r(x) \in \mathfrak{g}_+$, $x_- = r^*(x) \in \mathfrak{g}_-$, where $x \in \mathfrak{g}^*$, rewrite this equality as

$$\langle [y_-, z_-], x\rangle + \langle [x_+, y_+], z\rangle + \langle [x_+, z_-], y\rangle = 0.$$

This is equivalent to

$$[x_+, z_-] = (x_+ \triangleright z)_- - (z_- \triangleright x)_+,$$

where $\triangleright = -\mathrm{ad}^*|_{\mathfrak{g}}$ stands for the conjugate to the adjoint representation. Since the mapping $i^* : \mathfrak{g}^* \to \mathfrak{g}_+^* \sim \mathfrak{g}_-$ is a homomorphism of $\mathfrak{g}_+$-modules (and the same is the case with replacement $\pm \to \mp$), the latter expression can be rewritten as

$$[x_+, z_-] = x_+ \triangleright z_- - z_- \triangleright x_+,$$

where $\triangleright$ is already considered as $-\mathrm{ad}^*|_{\mathfrak{g}_\pm}$. But this is exactly the cross-commutator in the classical double $\mathcal{D}(\mathfrak{g}_+)$ [5]. □

Theorem 2.2 is the quasiclassical analog of the theorem proven in [14] for Hopf algebras (strictly speaking, finite dimensional). Recall that a Hopf algebra $\mathcal{A}$ is called quasitriangular [5] if there exists an invertible element $\mathcal{R} \in \mathcal{A}^{\otimes 2}$ (the universal R-matrix), such that

$$(\Delta \otimes \mathrm{id})(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{23}, \qquad (\mathrm{id} \otimes \Delta)(\mathcal{R}) = \mathcal{R}_{13}\mathcal{R}_{12},$$
$$\mathcal{R}\Delta(a) = \Delta'(a)\mathcal{R}, \quad a \in \mathcal{A},$$

where the prime denotes the opposite coproduct. These conditions imply that $\mathcal{R}$ satisfies the Yang–Baxter equation

$$\mathcal{R}_{12}\mathcal{R}_{13}\mathcal{R}_{23} = \mathcal{R}_{23}\mathcal{R}_{13}\mathcal{R}_{12}.$$

The universal R-matrix defines two algebra and anti-coalgebra homomorphisms from $\mathcal{A}^*$ to $\mathcal{A}$, $a \to \langle a \otimes \mathrm{id}, \mathcal{R}\rangle$ and $a \to \langle \mathrm{id} \otimes a, \mathcal{R}^{-1}\rangle = \langle \mathrm{id} \otimes a, (S \otimes \mathrm{id})(\mathcal{R})\rangle$, their images denoted $\mathcal{A}_+$ and $\mathcal{A}_-$. Hopf algebra $\mathcal{A}_-$ is isomorphic to $\mathcal{A}_+^*$ taken with the opposite multiplication. Recall that the quantum double $\mathcal{D}(\mathcal{A}_+)$, which is built on the tensor product of $\mathcal{A}_+$ and $\mathcal{A}_{+,\mathrm{op}}^*$ embedded there as sub-bialgebras [5]. The commutation relations between elements of these two factors are encoded in the Yang–Baxter equation for the canonical element $l = a^i \otimes a_i \in \mathcal{A}_{+,\mathrm{op}}^* \otimes \mathcal{A}_+$, where $\{a_i\}$ is the basis in $\mathcal{A}_+$ and $\{a^i\}$ its dual in $\mathcal{A}_+^*$.

Theorem 2.3 (Radford [14]). *Embeddings $\mathcal{A}_\pm \to \mathcal{A}$ define a homomorphism $\mathcal{D}(\mathcal{A}_+) \to \mathcal{A}$ of Hopf algebras. The universal R-matrix $\mathcal{R} \in \mathcal{A}^{\otimes 2}$ is the image of the canonical element $l \in \mathcal{D}^{\otimes 2}(\mathcal{A}_+)$ under this homomorphism.*

Proof. The map $\mathcal{D}(\mathcal{A}_+) \to \mathcal{A}$ is defined as identical on $\mathcal{A}_+ \otimes 1$ and the isomorphism $1 \otimes \mathcal{A}_{+,\mathrm{op}}^* \to \mathcal{A}_-$ given by the universal R-matrix. It respects the bialgebra structures when restricted to these sub-bialgebras. The image of the canonical element l under this mapping is the R-matrix, by Lemma 2.1, and the cross-relations in the quantum double go over into the quantum Yang–Baxter equation fulfilled by the R-matrix. Hence the map of concern is a homomorphism. The image of this homomorphism includes sub-Hopf algebras $\mathcal{A}_\pm$ and it is exactly that subalgebra in $\mathcal{A}$ where the R-matrix is supported. □

Remark 2.4. In the Drinfeld–Jimbo semi-simple Lie bialgebras [5], the subspaces $\mathfrak{g}_+$ and $\mathfrak{g}_-$ intersect by the Cartan subalgebra. The corresponding quantized universal enveloping algebras belong to the class of the factorizable Hopf algebras introduced in [19]. For that type of Hopf algebras, Theorem 2.3 was proven therein. A different class of Lie bialgebras are those with skew-symmetric r-matrices. There, the Lie subalgebra $\mathfrak{g}_+$ coincides with $\mathfrak{g}_-$. The simplest example of that kind is the coboundary Borel subalgebra $b(2)$ in sl(2). The double of quantized $\mathcal{U}(b(2))$ was studied in [20,21].

3. Inhomogeneous Lie bialgebras and their quantization

Let $(\mathfrak{g}, \mathfrak{g}^*)$ be a Lie bialgebra endowed with an involutive map $\sigma\colon \mathfrak{g} \to \mathfrak{g}$. We assume σ to be automorphism of $\mathfrak{g}$, while the dual map σ^* an anti-automorphism of $\mathfrak{g}^*$.

Definition 3.1. A triple $(\mathfrak{g}, \mathfrak{g}^*, \sigma)$ such that σ- and σ^*-invariant subspaces are Lie subalgebras in $\mathfrak{g}$ and $\mathfrak{g}^*$ is called inhomogeneous Lie bialgebra.

It follows from the definition that $\mathfrak{g}$ is represented by the semi-direct sum $\mathfrak{g} = \mathfrak{h} \triangleright \mathfrak{v}$, where $\mathfrak{v}$ is the eigenspace of σ corresponding the eigenvalue -1. Since $\mathfrak{v}$ is assumed to be a Lie subalgebra, it is commutative. The Lie subalgebra $\mathfrak{h}$ is the subspace of σ-stable points. The dual Lie algebra $\mathfrak{g}^* = \mathfrak{h}^* \triangleleft \mathfrak{v}^*$ has the same structure as $\mathfrak{g}$, with σ replaced by $-\sigma^*$. There, the subalgebra $\mathfrak{h}^*$ is a commutative ideal. In terms of the basis elements $H_i \in \mathfrak{h}$ and $X^\mu \in \mathfrak{v}$, the Lie bracket and cobracket on $\mathfrak{g}$ read

$$\begin{aligned} &\delta(X^\mu) = \gamma^\mu_{\rho\sigma}(X^\rho \otimes X^\sigma), \qquad \delta(H_i) = \alpha^k_{\rho i}(X^\rho \otimes H_k - H_k \otimes X^\rho), \\ &[X^\mu, X^\nu] = 0, \qquad [H_i, H_k] = C^m_{ik} H_m, \qquad [H_i, X^\mu] = A^\mu_{i\nu} X^\nu. \end{aligned} \tag{1}$$

The summation over repeating indices is understood throughout the paper. The tensors C^m_{ik} and $\gamma^\mu_{\rho\sigma}$ are the structure constants of Lie algebras $\mathfrak{h}$ and $\mathfrak{v}^*$, correspondingly. Matrices A_i and α_μ realize representations of $\mathfrak{h}$ on $\mathfrak{v}$ and $\mathfrak{v}^*$ on $\mathfrak{h}^*$, respectively.

To form a Lie bialgebra, the Lie structures on $\mathfrak{g}$ and $\mathfrak{g}^*$ must be compatible [5]. For the inhomogeneous Lie bialgebras that condition takes the form

$$A^\mu_{i\nu}\gamma^\nu_{\rho\sigma} - \gamma^\mu_{\nu\sigma}A^\nu_{i\rho} - \gamma^\mu_{\rho\nu}A^\nu_{i\sigma} = A^\mu_{k\sigma}\alpha^k_{\rho i} - A^\mu_{k\rho}\alpha^k_{\sigma i}, \tag{2}$$

$$\alpha^k_{\mu m}C^m_{ij} - C^k_{im}\alpha^m_{\mu j} - C^k_{mj}\alpha^m_{\mu i} = \alpha^k_{\nu j}A^\nu_{i\mu} - \alpha^k_{\nu i}A^\nu_{j\mu}. \tag{3}$$

The inhomogeneous Lie bialgebras form a category which we denote as $\mathcal{B}$. Its morphisms respect Lie brackets and cobrackets and commute with the involution σ.

Remark 3.2. It will be shown in the next section that there is a Lie algebra structure on the linear sum $\mathfrak{h} + \mathfrak{v}^*$ including $\mathfrak{h}$ and $\mathfrak{v}^*$ as subalgebras. Conversely, given a decomposition of a Lie algebra into the sum of Lie subalgebras $\mathfrak{a}$ and $\mathfrak{b}$, one has actions of $\mathfrak{a}$ on $\mathfrak{b}$ and vice versa. An inhomogeneous Lie bialgebra can be built by setting $\mathfrak{h} = \mathfrak{a}$, $\mathfrak{v} = \mathfrak{b}^*$, where $\mathfrak{b}^*$ is assumed to be abelian, and $\mathfrak{a}$ acts on $\mathfrak{b}^*$ by the dual conjugate action. Inhomogeneous Lie bialgebras are in one-to-one correspondence with such decompositions, which are the Lie algebra counterparts of the matched pairs of groups [4].

Denote H, H^*, V, V^* and $G = H \triangleright V$, $G^* = H^* \triangleleft V^*$, the simply connected Lie groups corresponding to the Lie algebras $\mathfrak{h}, \mathfrak{h}^*, \mathfrak{v}, \mathfrak{v}^*$ and $\mathfrak{g}, \mathfrak{g}^*$. Our quantization method relies on the quantum duality principle [5,22] as applied to the problem of "exponentiating" Lie bialgebras of concern. Following this principle, we consider a quantized enveloping algebra as a set of non-commutative functions on the group G^*. In accordance with the dual group method [23], we are looking for the deformed coproduct on the quantized $\mathcal{U}(\mathfrak{g})$ in the same form as if the generators of $\mathfrak{g}$ were commutative. In that commutative case, the coproduct is given by the same formulas as on the function algebra on the dual group G^*, in a neighborhood of the identity parameterized by the Lie algebra $\mathfrak{g}^*$. The problem is then reduced to finding commutation relations among the generators such that the coproduct would be coassociative and homomorphic. Integrating the Lie product in $\mathfrak{g}^*$ to the multiplication in G^*, we fix the coproduct on the generators:

$$\Delta(X^\mu) = D^\mu(X \otimes 1, 1 \otimes X), \qquad \Delta(H_i) = (e^{\alpha\cdot X})_i^k \otimes H_k + H_i \otimes 1. \tag{4}$$

We use the notation $D(\cdot,\cdot)$ for the Campbell–Hausdorff series corresponding to the Lie algebra $\mathfrak{v}^*$ and $\alpha \cdot X$ for the matrix with entries $\alpha^i_{\rho k} X^\rho$. The coproduct is evidently coassociative, as the elements X^μ commute with each other. The problem boils down to evaluating the complete set of quantum commutation relations consistent with (4). We will search for them in the form

$$[X^\mu, X^\nu] = 0, \qquad [H_i, H_k] = C(X)^m_{ik} H_m, \qquad [H_i, X^\mu] = A(X)^\mu_i, \tag{5}$$

treating the quantized structure coefficients as formal series in elements X^μ.

Theorem 3.3. *There exists the unique quantization of the bialgebra* $(\mathfrak{g}, \mathfrak{g}^*)$ *as a Hopf algebra with coproduct* (4) *and commutation relations* (5), *such that*

$$C(0)^m_{ik} = C^m_{ik}, \qquad \frac{\partial A^\mu_i}{\partial X^\nu}(0) = A^\mu_{i\nu}.$$

This quantization is a functor from the category $\mathcal{B}$ to a subcategory of Hopf algebras.

Proof. First, one has to check the consistency between commutation relations (5) and coproduct (4). Substituting (4) into $[\Delta(H), \Delta(X)] = \Delta([H, X])$, we come to the equation

$$A(D(X', X''))^\mu_i = (e^{\alpha\cdot X'})^k_i \partial''_\nu D^\mu(X', X'') A(X'')^\nu_k + \partial'_\nu D^\mu(X', X'') A(X')^\nu_i, \tag{6}$$

where primes distinguish the tensor factors. Regarding X^μ as the coordinate functions on the Lie group V^*, we consider $A(X)^\mu_i$ as a set of vector fields labeled by the index i, in a normal neighborhood of the identity. Then, Eq. (6) is nothing else than

$$A(\xi \circ \zeta)^\mu_i = (e^{\alpha\cdot\xi})^k_i L_\xi A(\zeta)^\mu_k + R_\zeta A(\xi)^\mu_i, \quad \xi, \zeta \in \mathfrak{v}^*, \tag{7}$$

where L_ξ, R_ξ stand for the left and right actions of the group V^* on the vector fields. Note that they both commute with the action via the matrices α_μ. Transition to the functions $\hat{A}(\xi) = R^{-1}_\xi A(\xi)$ leads to the group 1-cocycle equation

$$\hat{A}(\xi \circ \zeta)^\mu_i = (e^{\alpha\cdot\xi})^k_i \,\mathrm{ad}(\xi) \hat{A}(\zeta)^\mu_k + \hat{A}(\xi)^\mu_i, \tag{8}$$

which has the unique solution, provided the differential $\mathrm{d}\hat{A}(0)$ is a corresponding 1-cocycle of the Lie algebra $\mathfrak{v}^*$. That is a part of the Lie bialgebra consistency conditions (2) on the pair $(\mathfrak{g}, \mathfrak{g}^*)$. The explicit formula for the functions $A(X)_i^\nu$ is

$$A(X)_i^\mu = \left(\frac{\gamma'' \cdot X}{e^{\gamma'' \cdot X} - 1} \frac{e^{\alpha' \cdot X + \gamma'' \cdot X} - 1}{\alpha' \cdot X + \gamma'' \cdot X} \right)_{i\nu}^{k\mu} A_{k\rho}^\nu X^\rho. \tag{9}$$

Here $(\gamma \cdot X)_\nu^\mu = \gamma_{\sigma\nu}^\mu X^\sigma$ specifies the adjoint representation of the Lie algebra $\mathfrak{v}^*$. We mark the matrices with primes to stress that they act on the different groups of indices. Note that formula (9) is simplified in the case of abelian $\mathfrak{v}^*$: then $A(X)_i^\mu$ takes the form $A(X)_i^\mu = ((e^{\alpha \cdot X} - 1)/\alpha \cdot X)_\nu^\mu A_{i\rho}^\nu X^\rho$.

Requirement $[\Delta(H), \Delta(H)] = \Delta([H, H])$ leads to the following two equations:

$$C(D(X', X''))_{ij}^k = C(X')_{ij}^k \tag{10}$$

meaning that the coefficients $C(X)_{jk}^i$ are actually constant, and

$$(e^{\alpha \cdot X})_m^k C_{ij}^m - C_{mn}^k (e^{\alpha \cdot X})_i^m (e^{\alpha \cdot X})_j^n = [H_i, (e^{\alpha \cdot X})_j^k] - [H_j, (e^{\alpha \cdot X})_i^k]. \tag{11}$$

The Lie algebra representation of $\mathfrak{v}^*$ on $\mathfrak{h}^*$ by the matrices α_μ induces an anti-homomorphism of the group V^* into the linear group $\mathrm{Aut}(\mathfrak{h})$. The expressions on the right-hand side of (11) are the vector fields $A(X)_i$ transferred by that map to $\mathrm{End}(\mathfrak{h})$. In terms of matrices $a = e^{\alpha \cdot X}$, we rewrite (11) as

$$a_m^k C_{ij}^m - C_{mn}^k a_i^m a_j^n = A(a)_{ij}^k - A(a)_{ji}^k \tag{12}$$

or

$$aC(a^{-1} \otimes a^{-1}) - C = A(a)\pi(a^{-1} \otimes a^{-1}), \tag{13}$$

where we introduced the anti-symmetrizer π: $\pi_{kl}^{ij} = \delta_k^i \delta_l^j - \delta_l^i \delta_k^j$. The left-hand side of this equation is a coboundary 1-cocycle on the linear group, so we must prove that for the right-hand side. Then, since group 1-cocycles are uniquely determined by their derivatives at the identity, (13) will follow from (3). By virtue of (7), we have

$$\begin{aligned} &A(ba)\pi((ba)^{-1} \otimes (ba)^{-1}) \\ &\quad = (A(b)(a \otimes a) + bA(a))(a^{-1} \otimes a^{-1})\pi(b^{-1} \otimes b^{-1}) \\ &\quad = A(b)\pi(b^{-1} \otimes b^{-1}) + b\{A(a)\pi(a^{-1} \otimes a^{-1})\}(b^{-1} \otimes b^{-1}) \end{aligned}$$

as required.

The counit is evident: $\epsilon(H_i) = \epsilon(X^\mu) = 0$. The antipode is determined on the generators by the coproduct: $S(X^\mu) = -X^\mu$, $S(H_i) = -(e^{-\alpha \cdot X})_i^k H_k$. Let us prove that it is extended anti-homomorphically over the whole algebra. That is trivial in the commutative X-sector. Further,

$$[S(H_i), S(X^\mu)] = (e^{-\alpha \cdot X})_i^k [H_k, X^\mu] = (e^{-\alpha \cdot X})_i^k A(X)_k^\mu = -A(-X)_i^\mu = S([X^\mu, H_i]),$$

as immediately follows from formula (9). Condition $[S(H_i), S(H_j)] = S([H_j, H_i])$ boils down to verification of

$$C^k_{mn} a_i^{-1^m} a_j^{-1^n} + a_i^{-1^m}[H_m, a_j^{-1^n}] + a_j^{-1^n}[a_i^{-1^m}, H_n] = a_m^{-1^k} C^m_{ij}, \quad a = e^{\alpha \cdot X}.$$

We represent it as

$$S(C^k_{mn} a_i^m a_j^n) + S([H_i, a_j^n]) + S([a_i^m, H_j]) = S(a_m^k C^m_{ij}),$$

which holds true in view of (12).

Thus, we described the Hopf structure of the quantized algebra $\mathcal{U}(\mathfrak{g})$, $\mathfrak{g} \in \mathcal{B}$. We have yet to check that the quantization is a natural transformation of categories. Let ϕ be a Lie bialgebra morphism $\mathfrak{g} \to \mathfrak{g}'$ such that $\phi\sigma = \sigma'\phi$. This implies that $\phi(\mathfrak{h}) \subset \mathfrak{h}'$ and $\phi(\mathfrak{v}) \subset \mathfrak{v}'$. We define the map $\Phi : \mathcal{U}_q(\mathfrak{g}) \to \mathcal{U}_q(\mathfrak{g}')$ by the same formulas on the generators as ϕ:

$$\Phi(X^\mu) = \phi^\mu_{\nu'} X^{\nu'}, \qquad \Phi(H_i) = \phi_i^{k'} H_{k'}.$$

The map Φ can be extended to the whole quantum algebras as a Hopf homomorphism. That is evident for the coproduct because it is given by the product in the dual Lie groups, and ϕ is a Lie bialgebra homomorphism. That can be shown for the commutation relations as well. Indeed, the quantum commutator (9) differs from the classical one by involvement of the matrices $(\alpha \cdot X)^i_k$ and $(\gamma \cdot X)^\mu_\nu$. They specify the adjoint representation of the subalgebra $\mathfrak{v}^* \subset \mathfrak{g}^*$. Because ϕ^* is a homomorphism of the dual Lie algebras and preserves σ-invariant subspaces, matrices $\phi_i^{k'}$ and $\phi^\nu_{\mu'}$ are pulled through $\alpha \cdot X$ and $\gamma \cdot X$ properly, e.g. $(\alpha_{\mu'} X^{\mu'})\phi = \phi(\alpha_\mu \phi(X^\mu))$, so the proof becomes immediate. □

Remark 3.4. We denote $\mathcal{U}_q(\mathfrak{g})$, the quantization of the universal enveloping algebra $\mathcal{U}(\mathfrak{g})$. Here q is a symbolical notation rather than a deformation parameter. The deformation parameter can be introduced by rescaling $\alpha \to \ln(q)\alpha$, $\gamma \to \ln(q)\gamma$ of the structure constants of $\mathfrak{g}^*$. As a module over $\mathbb{C}[[h]]$, $h = \ln(q)$, the algebra $\mathcal{U}_q(\mathfrak{g})$ is flat. The parameter h may take any value, while the conventional deformation quantization is defined as an algebra over formal series in h. In that respect, the algebra $\mathcal{U}_q(\mathfrak{g})$ is a particular case of deformation quantization.

Algebra $\mathcal{U}_q(\mathfrak{g})$ contains two classical objects: universal enveloping algebra $\mathcal{U}(\mathfrak{h})$ and the commutative algebra of functions $\mathrm{Fun}(V^*)$ on the group V^*. In accordance with our convention, we may assume $\mathrm{Fun}(V^*) \sim \mathcal{U}_q(\mathfrak{v})$. Actually, $\mathcal{U}_q(\mathfrak{g})$ is a bicrossproduct Hopf algebra $\mathcal{U}(\mathfrak{h}) \triangleright \mathcal{U}_q(\mathfrak{v})$, with the coaction on $\mathcal{U}(\mathfrak{h})$ given by $H_i \to (e^{\alpha \cdot X})^k_i \otimes H_k$.

4. Duality, double, and quantization

We denote $\mathcal{H}$ the image of the category $\mathcal{B}$ with respect to the quantization functor. The category $\mathcal{B}$ is evidently self-dual, the involution σ going over into $-\sigma^*$. Let us prove the analogous assertion for $\mathcal{H}$ and deduce explicitly the canonical element.

Theorem 4.1. *The category $\mathcal{H}$ is self-dual. Moreover, $\mathcal{U}_q^*(\mathfrak{g}) = \mathcal{U}_q(\mathfrak{g}^*)$.*

Proof. As a linear space, $\mathcal{U}_q(\mathfrak{g})$ is the tensor product $\mathcal{U}_q(\mathfrak{v}) \otimes \mathcal{U}(\mathfrak{h})$. There are two natural algebra maps from $\mathcal{U}^*(\mathfrak{h})$ and $\mathcal{U}(\mathfrak{v}^*)$ into $\mathcal{U}_q^*(\mathfrak{g})$: we set $\eta \to \varepsilon_{\mathfrak{v}} \otimes \eta$ and $\zeta \to \zeta \otimes \varepsilon_{\mathfrak{h}}$ for $\eta \in \mathcal{U}^*(\mathfrak{h})$ and $\zeta \in \mathcal{U}^*(\mathfrak{v})$. An easy check shows that

$$\langle \eta\zeta, \varphi(X)\psi(H)\rangle = \langle \eta \otimes \zeta, \varphi(\Delta(X))\psi(\Delta(H))\rangle = \langle \eta, \psi(H)\rangle\langle \zeta, \varphi(X)\rangle. \tag{14}$$

Here, we identify the functionals ζ, η with their images in $\mathcal{U}_q^*(\mathfrak{g})$. Formula (14) implies that linear spaces $\mathcal{U}^*(\mathfrak{h})$ and $\mathcal{U}(\mathfrak{v}^*)$ are isomorphically embedded into $\mathcal{U}_q^*(\mathfrak{g})$ (in fact, these are homomorphisms of associative algebras, see Appendix A), and the induced map $\mathcal{U}(\mathfrak{v}^*) \otimes \mathcal{U}^*(\mathfrak{h}) \to \mathcal{U}^*(\mathfrak{h})\mathcal{U}(\mathfrak{v}^*)$ is a linear bijection on $\mathcal{U}_q^*(\mathfrak{g})$.

Let us choose, as the generators of $\mathcal{U}_q^*(\mathfrak{g})$, the bases $\{\eta^i\} \subset \mathfrak{h}^*$ and $\{\zeta_\mu\} \subset \mathfrak{v}^*$ dual to $\{H_i\}$ and $\{X^\mu\}$. It can be shown that the coproduct and the commutation relations have the structure of (4) and (1), in terms of the generators ζ_μ, η^i. That is done in Appendix A. According to Theorem 3.3, the Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$ admits the unique quantization $\mathcal{U}_q(\mathfrak{g}^*)$ belonging to the category $\mathcal{H}$. Hence, it coincides with $\mathcal{U}_q^*(\mathfrak{g})$. □

Because of factorization (14) and due to the fact that the matrix elements $\langle \eta, \psi(H)\rangle$ and $\langle \zeta, \varphi(X)\rangle$ are the same as if H_i and ζ_μ were primitive elements of the classical universal enveloping algebras, we can write down the canonical element $\mathcal{T} \in \mathcal{U}_q(\mathfrak{g}^*) \otimes \mathcal{U}_q(\mathfrak{g})$. However, it is convenient to deal with the opposite algebra $\mathcal{U}_q(\mathfrak{g}^*)_{\mathrm{op}}$; moreover, it is that algebra which takes part in construction of the double, the subject of our further interest. In Appendix A, we derive the formula

$$\mathcal{T} = \exp(\zeta_\mu \otimes X^\mu)\exp(\eta^i \otimes H_i) \tag{15}$$

for the canonical element $\mathcal{T} \in \mathcal{U}_q(\mathfrak{g}^*)_{\mathrm{op}} \otimes \mathcal{U}_q(\mathfrak{g})$.

Now, we proceed to the study of the double in the category $\mathcal{H}$. Written explicitly, the Lie product in $\mathcal{D}(\mathfrak{g})$ is

$$\begin{aligned}
&[X^\mu, X^\nu] = 0, \qquad [\eta^i, \eta^j] = 0, \qquad [X^\mu, \eta^i] = 0, \qquad [H_i, H_k] = C_{ik}^m H_m,\\
&[\zeta_\mu, \zeta_\nu] = \gamma_{\nu\mu}^\sigma \zeta_\sigma, \qquad [H_i, \zeta_\mu] = -\alpha_{\mu i}^k H_k - A_{i\mu}^\nu \zeta_\nu, \qquad [H_i, X^\mu] = A_{i\nu}^\mu X^\nu,\\
&[H_i, \eta^j] = C_{ki}^j \eta^k + \alpha_{\mu i}^j X^\mu, \qquad [\zeta_\mu, \eta^i] = -\alpha_{\mu k}^i \eta^k,\\
&[\zeta_\mu, X^\nu] = -\gamma_{\sigma\mu}^\nu X^\sigma - A_{i\mu}^\nu \eta^i.
\end{aligned} \tag{16}$$

It is seen that the linear sum $\mathfrak{h} + \mathfrak{v}^*$ forms a Lie subalgebra (cf. Remark 3.2) acting on the abelian ideal $\mathfrak{h}^* + \mathfrak{v}$. Observe that the classical double of $\mathfrak{g} \in \mathcal{B}$, is again a Lie bialgebra from $\mathcal{B}$. The Lie coalgebra on $\mathcal{D}(\mathfrak{g})$ is that of the direct sum $\mathfrak{g} \oplus \mathfrak{g}^*$, and the corresponding involution is $\sigma \oplus (-\sigma^*)$.

Theorem 4.2. *The quantum double construction preserves the category $\mathcal{H}$. Moreover, the classical and quantum doubles are related by the formula $\mathcal{D}(\mathcal{U}_q(\mathfrak{g})) = \mathcal{U}_q(\mathcal{D}(\mathfrak{g}))$.*

Proof. As a coalgebra, the quantum double coincides with the tensor product of $\mathcal{U}_q(\mathfrak{g})$ and $\mathcal{U}_q(\mathfrak{g}^*)_{\mathrm{op}}$, which are embedded as subalgebras, at the same time. Therefore, to prove the theorem, it suffices to show that the cross-relations have the appropriate form. Then, we will satisfy the conditions of Theorem 3.3 which states the uniqueness of the quantization and provides its explicit form. The cross-relations are deduced from the Yang–Baxter equation on the canonical element in the double and are written as

$$e_\mu e^\nu = e^\alpha e_\beta m^\nu_{\gamma\alpha\sigma} m^{\rho\beta\sigma}_\mu S^\gamma_\rho,$$

where $\{e_\mu\}$ is a linear basis in $\mathcal{U}_q(\mathfrak{g})$ and $\{e^\mu\} \subset \mathcal{U}_q(\mathfrak{g}^*)$ its dual. The tensors $m^\nu_{\gamma\alpha\sigma}$ and $m^{\rho\beta\sigma}_\mu$ denote the iterated coproduct structure constants, and S^γ_ρ is the matrix of the antipode. Using explicit formulas (4) for the coproducts

$$(\Delta \otimes \mathrm{id}) \circ \Delta(X) = D(X \otimes 1 \otimes 1, 1 \otimes X \otimes 1, 1 \otimes 1 \otimes X),$$

$$(\Delta \otimes \mathrm{id}) \circ \Delta(H) = e^{\alpha \cdot X} \otimes e^{\alpha \cdot X} \otimes H + e^{\alpha \cdot X} \otimes H \otimes 1 + H \otimes 1 \otimes 1,$$

$$(\Delta \otimes \mathrm{id}) \circ \Delta(\eta) = D(\eta \otimes 1 \otimes 1, 1 \otimes \eta \otimes 1, 1 \otimes 1 \otimes \eta),$$

$$(\Delta \otimes \mathrm{id}) \circ \Delta(\zeta) = e^{A \cdot \eta} \otimes e^{A \cdot \eta} \otimes \zeta + e^{A \cdot \eta} \otimes \zeta \otimes 1 + \zeta \otimes 1 \otimes 1, \tag{17}$$

we get the required result. Consider, e.g.,

$$H\zeta = \langle e^{-\alpha \cdot X}, \zeta \rangle H + \langle e^{-\alpha \cdot X}, e^{A \cdot \eta} \rangle \zeta H + \langle H, e^{-A \cdot \eta} \rangle \zeta$$

(only non-vanishing terms retained). Thus, we obtain the expression for the commutator $[H_i, \zeta_\mu] = -\alpha^k_{\mu i} H_k - A^\nu_{i\mu}\zeta_\nu$. Similarly, one can prove that X^μ and η^i form a commutative algebra, invariant under the adjoint action of H_i and ζ_μ. □

Let us illustrate the construction of the double by the example when $\mathfrak{g} = \mathfrak{h}$ with the nil cobracket. In this case, $\mathcal{U}_q(\mathfrak{g}) = \mathcal{U}(\mathfrak{g})$, and the quantum double is that of the classical universal enveloping algebra. The double of the group algebra of a compact group finds applications in topological filed theories with topological interactions (see, e.g. [24] and references therein). The algebra $\mathcal{D}(\mathcal{U}(\mathfrak{g}))$ is generated by the elements H_i and η^i with the undeformed commutation relations. The universal R-matrix is then the group element $\mathcal{R} = \exp(\eta^i \otimes H_i)$. Condition $\mathcal{R}\Delta(H_i) = \Delta'(H_i)\mathcal{R}$ immediately follows from the commutation relations, because the group element is invariant with respect to the adjoint action of the subalgebra $\mathfrak{h}$. Equation $\mathcal{R}\Delta(\eta^i) = \Delta'(\eta^i)\mathcal{R}$ is better to check in the form

$$\mathcal{R}_{12}(\Delta \otimes \mathrm{id})(\exp(\eta^i \otimes H_i))\mathcal{R}^{-1}_{12} = (\Delta' \otimes \mathrm{id})(\exp(\eta^i \otimes H_i)),$$

which is just the Yang–Baxter equation. It is also a simple corollary of the $\mathfrak{h}$-invariance of the R-matrix. We use here the fact that the group element $\exp(\eta^i \otimes H_i)$ is a bicharacter of the double.

The remainder of this section is devoted to quantization of coboundary Lie bialgebras from $\mathcal{B}$. To use the advantages of functorial property of the quantization, we assume that the r-matrix, as a Lie bialgebra morphism $\mathfrak{g}^* \to \mathfrak{g}$ (with $\mathfrak{g}^*$ equipped with the opposite bracket) be that in $\mathcal{B}$. It means that r commutes with the involution σ (we recall that for dual Lie bialgebra $(\mathfrak{g}^*, \mathfrak{g})$, the involutive map is taken to be $-\sigma^*$). The general form of r is then

$$r = \mathcal{P}^i_\mu H_i \otimes X^\mu + \mathcal{Q}^i_\mu X^\mu \otimes H_i. \tag{18}$$

Proposition 4.3. *The universal R-matrix $\mathcal{R} \in \mathcal{U}_q^{\otimes 2}(\mathfrak{g})$ quantizing the classical r-matrix* (18) *is given by the formula*

$$\mathcal{R} = \exp(\mathcal{P}^i_\mu H_i \otimes X^\mu) \exp(\mathcal{Q}^i_\mu X^\mu \otimes H_i). \tag{19}$$

Proof. The r-matrix defines a homomorphism $\mathcal{D}(\mathfrak{g}_+) \to \mathfrak{g}$ according to Theorem 2.2. We apply the quantization functor to this homomorphism getting a homomorphism of Hopf algebras $\mathcal{D}(\mathcal{U}_q(\mathfrak{g}_+)) \to \mathcal{U}_q(\mathfrak{g})$, along the line of Theorem 3.3. Then, we evaluate that Hopf algebra homomorphism on the canonical element (15) thus coming to (19). □

Formula (19) generalizes the expression for the universal R-matrix corresponding to the jordanian quantization $\mathcal{U}(b(2))$ of the Borel subalgebra $b(2) \subset \mathrm{sl}(2)$ [20].

5. Conclusion

Drinfeld's conjecture of the possibility to quantize an arbitrary Lie bialgebra was proven by Etingof and Kazhdan [25]. Although there are numerous examples of quantum algebras, the problem of exponentiating a Lie bialgebra structure in every particular case remains highly non-trivial. In this paper, we did it for a class of algebras playing significant role in the classical differential geometry and physics, namely inhomogeneous Lie algebras. This class forms a nice self-dual category invariant under the quantum double operation. The quantization built is a functor from the Lie bialgebra category of concern into the sub-category of Hopf algebras. This functor commutes with the functor of dualization and the double procedure. We showed that the quantization of inhomogeneous Lie algebras possessing classical r-matrix contains a quasitriangular Hopf algebra and gave an explicit expression for the universal R-matrix.

The class of Hopf algebras considered in this paper includes the twisted universal enveloping Lie algebras investigated in [10–12]. Those are characterized by the identification $\mathfrak{v} \sim \mathfrak{h}^*$ and involve non-degenerate Lie algebra 1-cocycles in building the semi-direct sum $\mathfrak{h} \triangleright \mathfrak{h}^*$. The class of inhomogeneous Lie bialgebras is much wider than that studied in [10–12], since the double of a triangular Hopf algebra is not a triangular one.

The appearance of Lie group 1-cocycles in construction of quantization is quite understandable. According to Drinfeld [5], a group 1-cocycle with value in the Lie algebra exterior square defines a Poisson–Lie structure on the group. Generators $H_i \in \mathfrak{h}$ and $X^\mu \in \mathfrak{v}$ are the coordinate functions on the dual group $G^* = H^* \triangleleft V^*$, and the quantum commutation relations among them represent nothing else than the Poisson bracket. Indeed, the classical commutation relations of the types $[H_i, H_j]$ and $[X^\mu, X^\nu]$ are given by the Poisson–Lie structure on the abelian subgroup H^* and the trivial Poisson bracket on the subgroup V^*. Further, the Poisson bracket must satisfy the relation

$$\begin{aligned}\Delta(\{H, X\}) &= \{e^{\alpha\cdot X} \otimes H + H \otimes 1, D(X \otimes 1, 1 \otimes X)\} \\ &= (e^{\alpha\cdot X} \otimes 1)\{1 \otimes H, D(X \otimes 1, 1 \otimes X)\} \\ &\quad + \{H \otimes 1, D(X \otimes 1, 1 \otimes X)\}\Delta(\{H, H\}) \\ &= \{e^{\alpha\cdot X} \otimes H + H \otimes 1, e^{\alpha\cdot X} \otimes H + H \otimes 1\} \\ &= e^{\alpha\cdot X}e^{\alpha\cdot X} \otimes \{H, H\} + \{H, H\} \otimes 1 + \{e^{\alpha\cdot X}, H\} \otimes H + \{H, e^{\alpha\cdot X}\} \otimes H \end{aligned} \tag{20}$$

(we drop all the indices for the reason of transparency). These expressions involve the Poisson bracket and the multiplication in the function algebra on the group $G^* = H^* \triangleleft V^*$. It is seen that only the product in Fun(V^*) actually enters formula (20). This product commutative and coincides with that on $\mathcal{U}_q(\mathfrak{v}) \subset \mathcal{U}_q(\mathfrak{g})$. So, the commutation relations (5) and the Lie–Poisson bracket among H_i and X^μ satisfy the same functional equations (see Theorem 3.3) and, therefore, must coincide.

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Appendix A

The aim of this appendix is to exhibit the details of the proof of the formula (15). The canonical element for the universal enveloping algebra $\mathcal{U}(\mathfrak{v}^*)$ is [5]

$$e^{\zeta_\mu \otimes X^\mu} = \sum_n \sum_{\vec{\mu}} (\zeta_{\mu_1}, \ldots, \zeta_{\mu_n}) \otimes X^{\mu_1}, \ldots, X^{\mu_n},$$

where $\vec{\mu} = \{\mu_1, \ldots, \mu_n\}$ stands for the ordered multi-index of length n, and the parentheses denote symmetrized monomials $(\zeta_{\mu_1}, \ldots, \zeta_{\mu_n}) = (1/n!)(1/s(\vec{\mu}))(\sum_\sigma \zeta_{\sigma(\mu_1)}, \ldots, \zeta_{\sigma(\mu_n)})$ of degree n. Here, σ runs over all permutations of the multi-index $\vec{\mu}$. The symmetry coefficient $s(\vec{\mu})$ is equal to the order of the stability subgroup of $\vec{\mu}$. Similarly, we have

$$e^{\eta^i \otimes H_i} = \sum_n \sum_{\vec{i}} \eta^{i_1}, \ldots, \eta^{i_n} \otimes (H_{i_1}, \ldots, H_{i_n})$$

for the algebra $\mathcal{U}(\mathfrak{h})$. Hence, due to the factorization of the matrix elements of the canonical pairing (see Section 4), the element $e^{\zeta_\mu \otimes X^\mu} e^{\eta^i \otimes H_i}$ is canonical for $\mathcal{U}_q(\mathfrak{g})^*_{\mathrm{op}}$. It remains to check formulas (4) and (5) for the coproduct and the commutation relations on the generators

ζ_μ, η^i.

$$\begin{aligned}&\langle \eta^i \eta^j, X^{\mu_1}, \dots, X^{\mu_n}(H_{i_1}, \dots, H_{i_m})\rangle\\&\quad = \varepsilon(X^{\mu_1}, \dots, X^{\mu_n})\langle \eta^i \otimes \eta^j, \Delta(H_{i_1}, \dots, H_{i_m})\rangle\\&\quad = \varepsilon(X^{\mu_1}, \dots, X^{\mu_n})\langle \eta^i \otimes \eta^j, \Delta_0(H_{i_1}, \dots, H_{i_m})\rangle.\end{aligned}$$

The transition from Δ to the classical coproduct Δ_0 is performed in the following way. While pushing the factors $(e^{\alpha\cdot X})_i^k$ appearing in $\Delta(H_{i_k})$ to the left, we act as though they commute with all H_i. That can be done because the elements X^μ's generate an ideal and, once appeared, they are annihilated by the elements η^i. Then, all the factors $(e^{\alpha\cdot X})_i^k$ being placed on the left are reduced to 1. So, the generators η^i have the classical commutative multiplication. Further,

$$\begin{aligned}&\langle \zeta_\mu \zeta_\nu, X^{\mu_1}, \dots, X^{\mu_n}(H_{i_1}, \dots, H_{i_m})\rangle\\&\quad = \langle \zeta_\mu \otimes \zeta_\nu, \Delta(X^{\mu_1}, \dots, X^{\mu_n}(H_{i_1}, \dots, H_{i_m}))\rangle\\&\quad = \langle \zeta_\mu \otimes \zeta_\nu, \Delta(X^{\mu_1}, \dots, X^{\mu_n})((H_{i_1}, \dots, H_{i_m}) \otimes 1)\rangle\\&\quad = \varepsilon(H_{i_1}, \dots, H_{i_m})\langle \zeta_\mu \otimes \zeta_\nu, \Delta(X^{\mu_1}, \dots, X^{\mu_n})\rangle,\end{aligned}$$

and, therefore, ζ's generate the classical universal enveloping Lie algebra $\mathcal{U}(\mathfrak{v}^*)$, with the ordinary Lie commutation relations. Among the matrix elements

$$\begin{aligned}&\langle \zeta_\mu \eta^i, X^{\mu_1}, \dots, X^{\mu_n}(H_{i_1}, \dots, H_{i_m})\rangle\\&\quad = \langle \zeta_\mu \otimes \eta^i, (X^{\mu_1}, \dots, X^{\mu_n} \otimes 1)\Delta(H_{i_1}, \dots, H_{i_m})\rangle\end{aligned}$$

only those survive, where $n \le 1$. Developing products of $\Delta(H)$'s, we see that monomials $1 \otimes H_{i_1}, \dots, H_{i_k}$ turn out to be symmetrized automatically, hence, we can retain terms of the first degree in $1 \otimes H_i$ only. Furthermore, if $n = 1$, then with necessity $m = 1$. The non-vanishing pairings are

$$\begin{aligned}\langle \zeta_\mu \eta^i, X^{\mu_1}(H_{i_1}, \dots, H_{i_m})\rangle &= \langle \zeta_\mu \otimes \eta^i, X^{\mu_1}(H_{i_1}, \dots, H_{i_{m-1}})(e^{\alpha\cdot X})_{i_m}^k \otimes H_k\rangle\\&= \langle \zeta_\mu \otimes \eta^i, \varphi(X) \otimes H_k\rangle,\end{aligned}$$

where $\varphi(X)$ is a result of pulling the exponential to the left. Thus, we state that the commutator $\zeta_\mu \eta^i - \eta^i \zeta_\mu$ does not vanish only on the elements $(H_{i_1}, \dots, H_{i_m})$, therefore, it depends solely on η^i.

We have yet to find the coproduct. It is straightforward that $\Delta(\eta)$ survives on the elements containing no X's and, therefore, is expressed by the Campbell–Hausdorff series corresponding to $\mathcal{U}(\mathfrak{h})$. For $\Delta(\zeta)$ the non-trivial pairing is with elements $X^\mu \otimes 1$ and $(H_{i_1}, \dots, H_{i_m}) \otimes X^\mu$. While pulling H's to the right, we can assume that they commute with X's via the classical relations. The commutator is linear in X because the higher degrees will vanish. Thus, we come to the desired formula

$$\Delta(\zeta_\mu) = (e^{A\cdot\eta})_\mu^\nu \otimes \zeta_\nu + \zeta_\mu \otimes 1.$$

Thus, we verified that the algebra generated by η^i, ζ_μ belongs to the category $\mathcal{H}$ and proved the formula (15) for the canonical element.

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